

Maximum Lifetime Coverage Problem

Input: S_1, \dots, S_n ; a set of targets that can be monitored by each sensor.

Output: Suppose that C_1, \dots, C_p are set of all target coverages.

Output: t_1, \dots, t_p : time we use each target coverage.

Constraint: All sensors must be used for less than 1 hour

$$\text{For all } j, \sum_{i \in C_j} t_i \leq 1$$

$$\sum_{i \in C_j} a_{j,i} t_i \leq 1 \quad \text{where } a_{j,i} = \begin{cases} 1 & i \in C_j \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{bmatrix} a_{1,1} & \dots & a_{1,p} \\ a_{2,1} & \dots & a_{2,p} \\ \vdots & & \vdots \\ a_{n,1} & \dots & a_{n,p} \end{bmatrix} \begin{bmatrix} t_1 \\ \vdots \\ t_p \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Objective Function:

$$\text{Maximize } \sum_i t_i = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}^T \begin{bmatrix} t_1 \\ \vdots \\ t_p \end{bmatrix}$$

$$\text{Maximize } \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}^T \cdot \begin{bmatrix} t_1 \\ \vdots \\ t_p \end{bmatrix}$$

$$\begin{bmatrix} a_{1,1} & \dots & a_{1,p} \\ \vdots & & \vdots \\ a_{n,1} & \dots & a_{n,p} \end{bmatrix} \begin{bmatrix} t_1 \\ \vdots \\ t_p \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad \text{Dual}$$

PPM

$$\text{Minimize } \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}^T \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$\begin{bmatrix} a_{1,1} & \dots & a_{n,1} \\ \vdots & & \vdots \\ a_{1,p} & \dots & a_{n,p} \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \geq \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\text{Minimize } y_1 + \dots + y_n$$

$$a_{1,1}y_1 + \dots + a_{n,1}y_n \geq 1$$

$$a_{1,p}y_1 + \dots + a_{n,p}y_n \geq 1$$

Gary-Konemann Framework

$$1: \quad x \leftarrow \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad y \leftarrow \begin{pmatrix} \delta \\ \delta \\ \vdots \\ \delta \end{pmatrix}$$

2: Choose r^* that minimizes $A_i = a_{1,i}y_1 + a_{2,i}y_2 + \dots + a_{n,i}y_n \rightarrow \text{Minimum Cost Target Coverage.}$

3: If $A_{r^*} \geq 1$, terminates

4: For all j such that $a_{j,r^*} = 1$, $y_j \leftarrow y_j(1+\theta)$

5: $\theta x_{r^*} \leftarrow x_{r^*} + \tau$

6: Go to Step 2

How to set T ?

$$T := \frac{\ln(1+\theta)}{\ln(1+\theta) - \ln\delta}$$

Theorem By the algorithm, the usages of all sensors are less than 1 hour.

Proof Sensor i is used for T hours

$\hookrightarrow i \in C_{j^*}$ where j^* is selected in the algorithm.

$\hookrightarrow j^* \text{ minimizes } a_{1,j^*} y_1 + a_{2,j^*} y_2 + \dots + a_{n,j^*} y_n$

\hookrightarrow We increase the value of y_q for all q such that $a_{q,j^*} = 1$

\hookrightarrow We increase the value of y_q if $y_q \in C_{j^*}$

\hookrightarrow We increase the value of y_i

times that sensor i is used for T hours = # times y_i is multiplied by $(1+\theta)$

Sensor i 's usage is Tm . $\iff y_i = \delta(1+\theta)^m$

Suppose that sensor i is used for m times.

Before the last update $y_i = \delta(1+\theta)^{m-1}$

$$\underbrace{a_{j^*,j^*} y_i}_2 \leq a_{1,j^*} y_1 + a_{2,j^*} y_2 + \dots + a_{n,j^*} y_n < 1$$

$$y_i < 1$$

$$\delta(1+\theta)^{m-1} < 1$$

$$(1+\theta)^{m-1} < 1/\delta$$

$$\ln((1+\theta)^{m-1}) < \ln 1/\delta$$

$$(m-1)[\ln(1+\theta)] < -\ln\delta$$

$$m-1 < \frac{-\ln\delta}{\ln(1+\theta)}$$

$$m < 1 + \frac{-\ln\delta}{\ln(1+\theta)} = \frac{\ln(1+\theta) - \ln\delta}{\ln(1+\theta)}$$

$$\text{Sensor usage is } Tm \leq \frac{\ln(1+\theta)}{\ln(1+\theta) - \ln\delta}, \frac{\ln(1+\theta)}{\ln(1+\theta) - \ln\delta} = 1 \text{ hour.}$$

Theorem Number of iterations $\leq \frac{\ln(1+\theta)}{\ln(1+\theta) - \ln\delta} \frac{n}{T} = \frac{n \cdot (\ln(1+\theta) - \ln\delta)}{\ln(1+\theta) - \ln\delta}$

Proof The sum of sensor capacity at the beginning = n hours

Each iteration will use at least T hours from the capacity sum.

$$\# \text{ iterations} \leq n/T$$

IV

$$\min_{z \in \mathbb{R}^n} z_1 + \dots + z_n = y_1 + \dots + y_n \iff y_1^* + \dots + y_n^* = OPT$$

$$\therefore OPT = \min_{z \in \mathbb{R}^n} \frac{z_1 + \dots + z_n}{f(z)} \quad \boxed{1}$$

Lemma #iterations $\geq \ln\left(\frac{1}{\theta}\right) \frac{OPT}{\theta}$

Proof Let $y_i(t)$ be a value of y_i at Iteration t , and let $\Psi(t) = \sum_{i=1}^n y_i(t)$.

Recall that we select coverage C_{ij} and multiply y_j with $(1-\theta)$ if $a_{j,i} = 1$.

$$\Psi(t+1) = \boxed{\text{terms that are not multiplied}} + (1-\theta) \cdot \boxed{\text{terms that are multiplied}}$$

$$= \boxed{\text{Terms that are not multiplied}} + \boxed{\text{Terms that are multiplied}} \quad \text{all terms } \Psi(t)$$

$$+ \theta \cdot \boxed{\text{terms that are multiplied}}$$

$$= \Psi(t) + \theta [a_{1,i} y_1(t) + \dots + a_{n,i} y_n(t)]$$

$$= \Psi(t) + \theta \min_i [a_{1,i} y_1(t) + \dots + a_{n,i} y_n(t)]$$

$$f((y_1(t), \dots, y_n(t))) \leq \frac{\Psi(t)}{OPT}$$

$$\Psi(t+1) = \Psi(t) + \theta \min_i [a_{1,i} y_1(t) + \dots + a_{n,i} y_n(t)] = \Psi(t) + \theta \cdot f((y_1(t), \dots, y_n(t)))$$

For by previous lemma,

$$\boxed{OPT = \min_{z \in \mathbb{R}^n} \frac{z_1 + \dots + z_n}{f(z)}} \leq \frac{\boxed{y_1(t) + \dots + y_n(t)}}{\boxed{f((y_1(t), \dots, y_n(t))})} \quad \boxed{\Psi(t)}$$

$$f((y_1(t), \dots, y_n(t))) \leq \frac{\Psi(t)}{OPT}$$

$$\Psi(t+1) \leq \Psi(t) + \theta \frac{\Psi(t)}{OPT} = \Psi(t) \left[1 + \theta \frac{1}{OPT} \right] \leq \Psi(t) e^{\theta/OPT} \quad 1+x \leq e^x$$

$$\begin{aligned} \Psi(t) &\leq e^{\theta/OPT \cdot t} \cdot \Psi(0) = e^{\theta/OPT \cdot t} [\boxed{y_1(0) + \dots + y_n(0)}] \\ &= e^{\theta/OPT \cdot t} \cdot [\delta^n] \end{aligned}$$

Suppose that #iterations is T .

If we know that $\Psi(T) = y_1(T) + \dots + y_n(T)$
 $\geq a_{1,i} y_1(T) + \dots + a_{n,i} y_n(T) \quad \text{for all } i.$

$$\geq 1$$

$$\Psi(T) \geq 1 \quad \Psi(T) \leq e^{\theta/OPT \cdot T} [\delta^n]$$

$$e^{\theta/OPT \cdot T} [\delta^n] \geq 1$$

$$e^{\theta/OPT \cdot T} \geq \Psi(T)$$

$$\ln[e^{\theta/OPT \cdot T}] \geq \ln\left(\frac{1}{\delta^n}\right)$$

$$\frac{\theta}{OPT} \cdot T \geq \ln\left(\frac{1}{\delta^n}\right)$$

$$T \geq \frac{OPT}{\theta} \ln\left(\frac{1}{\delta^n}\right)$$

$\boxed{11}$

How to assyn ϵ ? $\zeta := (1+\theta)(1+\theta)n^{-\frac{1}{\theta}}$

Theorem #iterations $\leq \frac{2/\theta}{\epsilon} n \cdot \ln n = O(n \ln n)$

Proof #iterations $\leq n \cdot \frac{\ln(1+\theta) - \ln \zeta}{\ln(1+\theta)}$

$$= n \cdot \frac{\ln(1+\theta) - \ln[(1+\theta)(1+\theta)n^{\frac{1}{\theta}}]}{\ln(1+\theta)} \\ = n \cdot \frac{\ln(1+\theta) - \ln(1+\theta) - \ln((1+\theta)n^{\frac{1}{\theta}})}{\ln(1+\theta) \cdot \theta}$$

$$= n \cdot \frac{n}{\theta} \frac{\ln(1+\theta)n}{\theta} = \frac{n}{\theta^2} (\ln(1+\theta) + \ln n) = \frac{1}{\theta^2} n \ln n$$

very small compared to $\ln n$

When θ is small,

$$\ln(1+\theta) \approx \theta$$

□

Lemma Let OPT be the longest valid schedule's length (optimal value of primal problem)
= smallest $y_1^{*} + \dots + y_n^{*}$ (optimal value of dual problem)

For any $z \in (z_1, \dots, z_n) \in \mathbb{R}_{\geq 0}^n$, let

$$f(z) = \min_{i \in \{1, \dots, p\}} [a_{1,i} z_1 + a_{2,i} z_2 + \dots + a_{n,i} z_n]$$

smallest left side of dual constraints
suppose to be ≥ 1

$$\text{Then, } \min_{z \in \mathbb{R}^n} \frac{z_1 + \dots + z_n}{f(z)} = OPT$$

Result

$$\min_{A^T \cdot y \leq 1} y_1 + \dots + y_n \quad \leftrightarrow \quad \min \left[\frac{z_1 + \dots + z_n}{f(z)} \right]$$

easy to conduct
math analysis
from here.

Proof

Suppose that the optimal solution of the dual program is

$$\begin{cases} a_{1,1} y_1^* + a_{2,1} y_2^* + \dots + a_{n,1} y_n^* \geq 1 \\ \vdots \\ a_{1,p} y_1^* + a_{2,p} y_2^* + \dots + a_{n,p} y_n^* \geq 1 \end{cases}$$

Minimum is 1

$$\begin{bmatrix} y_1^* \\ \vdots \\ y_n^* \end{bmatrix} \quad OPT = y_1^* + \dots + y_n^*$$

$$f((y_1^*, \dots, y_n^*)) = 1$$

$$\min_{z \in \mathbb{R}^n} \frac{z_1 + \dots + z_n}{f(z)} \leq \frac{y_1^* + \dots + y_n^*}{f((y_1^*, \dots, y_n^*))} = y_1^* + \dots + y_n^* = OPT$$

Hence, $y_i = z_i / f(z)$ for all i .

$$\min_{i \in \{1, \dots, p\}} [a_{1,i} y_1 + a_{2,i} y_2 + \dots + a_{n,i} y_n] = \min_{i \in \{1, \dots, p\}} \left[a_{1,i} \frac{z_1}{f(z)} + a_{2,i} \frac{z_2}{f(z)} + \dots + a_{n,i} \frac{z_n}{f(z)} \right] \\ = \frac{1}{f(z)} \min_{i \in \{1, \dots, p\}} [a_{1,i} z_1 + a_{2,i} z_2 + \dots + a_{n,i} z_n] \\ = 1$$

$\therefore \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ is a possible solution of the dual program

At each iteration, the schedule is expanded by \mathcal{T} .

$$\# \text{iterations} \geq \ln\left(\frac{1}{n\delta}\right) \frac{\text{OPT}}{\theta} \implies \text{schedule length} \geq \boxed{\ln\left(\frac{1}{n\delta}\right) \frac{\text{OPT}}{\theta} \cdot \mathcal{T}}$$

$$\begin{aligned} \ln\left(\frac{1}{n\delta}\right) \cdot \frac{\mathcal{T}}{\theta} &= \ln\left(\frac{1}{n\delta}\right) \cdot \frac{\ln(1+\theta)}{\ln(1+\theta) - \ln n} \cdot \frac{1}{\theta} \\ &= \ln\left(\frac{1}{n(1+\theta)(1+\theta)n^{-1/\theta}}\right) \cdot \frac{\ln(1+\theta)}{\ln(1+\theta) - \ln(1+\theta) + \cancel{\frac{1}{\theta}} \ln(1+\theta)n^{-1/\theta}} \cdot \frac{1}{\theta} \\ &= \ln\left[\left(n(1+\theta)\right)^{1/\theta-1}\right] \cdot \frac{\ln(1+\theta)}{\cancel{\ln(1+\theta) - \ln(1+\theta) + \frac{1}{\theta} \ln(1+\theta)n^{-1/\theta}}} \cdot \cancel{\frac{1}{\theta}} \\ &= (1-\theta) \left[\ln(n(1+\theta)) \right] \cdot \frac{\ln(1+\theta)}{\cancel{\ln((1+\theta)n)}} = \frac{1-\theta}{\theta} \cdot \ln(1+\theta) \geq \frac{1-\theta}{1+\theta} \end{aligned}$$

try by yourself

$$\text{Schedule length} = \left(\frac{1-\theta}{1+\theta}\right) \cdot \text{OPT}$$

\uparrow
close to 1 when $\theta \rightarrow 0$

$$\# \text{iteration} = \frac{1}{\theta} n \log n$$

\downarrow
large when $\theta \rightarrow 0$

1. Trade-off between quality of results and computation time

2. When we have infinite computation time ($\theta=0$), we can obtain the optimal result!



Polynomial-Time Approximation Scheme (PTAS)